参考: ベアストウ・ヒチコック法の導出

$$
\begin{aligned}
f(x) &= a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n \\
&= (x^2+px+q)(b_0x^{n-2} + b_1x^{n-3} + \cdots + b_{n-3}x + b_{n-2}) + rx + s \\
&\text{係数比較して} \\
&a_0 = b_0 \\
&a_1 = b_1 + pb_0 \\
&a_k = b_k + pb_{k-1} + qb_{k-2}, (1 < k < n-1) \qquad (1) \\
&a_{n-1} = r + pb_{n-2} + qb_{n-3} \qquad (2) \\
&a_n = s + qb_{n-2} \qquad (3)
\end{aligned}
$$

式 (1) より

$$b_k = a_k - pb_{k-1} - qb_{k-2}(k \geq 2), b_{-1} = b_{-2} = 0 \qquad (4)$$

式 (4) を式 (2)(3) に代入すると

$$r = a_{n-1} - pb_{n-2} - qb_{n-3} = b_{n-1} \qquad (5)$$

$$s = a_n - qb_{n-2} = b_n + pb_{n-1} \qquad (6)$$

式 (5)(6) より、r, s は p, q の関数であることがわかるので、$r = r(p,q), s = s(p,q)$ とおく。$r = r(p,q) = 0, s = s(p,q) = 0$ となれば、$f(x)$ が (x^2+px+q) で因数分解できることになる。よって $r(p+\Delta p, q+\Delta q) = 0, s(p+\Delta p, q+\Delta q) = 0$ となる p, q を求める。この式を展開する。

$$r(p+\Delta p, q+\Delta q) = r(p,q) + \frac{\partial r}{\partial p}\Delta p + \frac{\partial r}{\partial q}\Delta q = 0 \qquad (7)$$

$$s(p+\Delta p, q+\Delta q) = s(p,q) + \frac{\partial s}{\partial p}\Delta p + \frac{\partial s}{\partial q}\Delta q = 0 \qquad (8)$$

$$\frac{\partial r}{\partial p} = \frac{\partial b_{n-1}}{\partial p}(\because Eq.(5)) \qquad (9)$$

$$\frac{\partial r}{\partial q} = \frac{\partial b_{n-1}}{\partial q}(\because Eq.(5)) \qquad (10)$$

$$\frac{\partial s}{\partial p} = \frac{\partial}{\partial p}(b_n + pb_{n-1}) = \frac{\partial b_n}{\partial p} + b_{n-1} + p\frac{\partial b_{n-1}}{\partial p}(\because Eq.(6)) \qquad (11)$$

$$\frac{\partial s}{\partial q} = \frac{\partial}{\partial q}(b_n + pb_{n-1}) = \frac{\partial b_n}{\partial q} + p\frac{\partial b_{n-1}}{\partial q}(\because Eq.(6)) \qquad (12)$$

式 (4) を p, q で偏微分する。

$$\frac{\partial b_k}{\partial p} = \frac{\partial}{\partial p}(a_k - pb_{k-1} - qb_{k-2}) = -b_{k-1} - p\frac{\partial b_{k-1}}{\partial p} - q\frac{\partial b_{k-2}}{\partial p} \qquad (13)$$

$$\frac{\partial b_k}{\partial q} = \frac{\partial}{\partial q}(a_k - pb_{k-1} - qb_{k-2}) = -p\frac{\partial b_{k-1}}{\partial q} - b_{k-2} - q\frac{\partial b_{k-2}}{\partial q} \qquad (14)$$

$k=0,1,\ldots$ と順に求めると

$$
\begin{aligned}
&\frac{\partial b_0}{\partial p}=-b_{-1}=0\\
&\frac{\partial b_1}{\partial p}=-b_0\\
&\frac{\partial b_2}{\partial p}=-b_1-p\frac{\partial b_1}{\partial p}=-b_1+pb_0\\
&\frac{\partial b_3}{\partial p}=-b_2-p\frac{\partial b_2}{\partial p}-q\frac{\partial b_1}{\partial p}=-b_2-p(-b_1+pb_0)-q(-b_0)\\
&\frac{\partial b_k}{\partial p}=-b_{k-1}-p\frac{\partial b_{k-1}}{\partial p}-q\frac{\partial b_{k-2}}{\partial p}=-b_{k-1}-p(-b_{k-2}+pb_{k-3})-q(-b_{k-3})
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\partial b_0}{\partial q}=0\\
&\frac{\partial b_1}{\partial q}=-p\frac{\partial b_0}{\partial q}-b_{-1}-q\frac{\partial b_{-1}}{\partial q}=0\\
&\frac{\partial b_2}{\partial q}=-p\frac{\partial b_1}{\partial q}-b_0-q\frac{\partial b_0}{\partial q}=-b_0\\
&\frac{\partial b_3}{\partial q}=-p\frac{\partial b_2}{\partial q}-b_1-q\frac{\partial b_1}{\partial q}=pb_0-b_1\\
&\frac{\partial b_4}{\partial q}=-p\frac{\partial b_3}{\partial q}-b_2-q\frac{\partial b_2}{\partial q}=-p(pb_0-b_1)-b_2-q(-b_0)\\
&\frac{\partial b_k}{\partial q}=-p\frac{\partial b_{k-1}}{\partial q}-b_{k-2}-q\frac{\partial b_{k-1}}{\partial q}=-p(pb_{k-4}-b_{k-3})-b_{k-2}-q(-b_{k-4})=\frac{\partial b_{k-1}}{\partial p}
\end{aligned}
$$

$\therefore\frac{\partial b_k}{\partial p}=\frac{\partial b_{k+1}}{\partial q}=-c_{k-1}$とおいて、式 (9)(10)(11)(12) に代入

$$
\begin{aligned}
&\frac{\partial r}{\partial p}=\frac{\partial b_{n-1}}{\partial p}=-c_{n-2}\\
&\frac{\partial r}{\partial q}=\frac{\partial b_{n-1}}{\partial q}=-c_{n-3}\\
&\frac{\partial s}{\partial p}=\frac{\partial b_n}{\partial p}+b_{n-1}+p\frac{\partial b_{n-1}}{\partial p}=-c_{n-1}+b_{n-1}-pc_{n-2}\\
&\frac{\partial s}{\partial q}=\frac{\partial b_n}{\partial q}+p\frac{\partial b_{n-1}}{\partial q}=-c_{n-2}-pc_{n-3}
\end{aligned}
$$

また、式 (13) に代入

$$
\begin{aligned}
\frac{\partial b_k}{\partial p}&=-b_{k-1}-p\frac{\partial b_{k-1}}{\partial p}-q\frac{\partial b_{k-2}}{\partial p}\\
&=-b_{k-1}-p(-c_{k-2})-q(-c_{k-3})=-c_{k-1} \qquad (15)
\end{aligned}
$$

式 (7) (8) に式 (5) (6) とともに代入

$$
\begin{aligned}
r(p+\mathit{\Delta} p, q+\mathit{\Delta} q) &= r(p,q) + \frac{\partial r}{\partial p}\mathit{\Delta} p + \frac{\partial r}{\partial q}\mathit{\Delta} q \\
&= b_{n-1} + (-c_{n-2})\mathit{\Delta} p + (-c_{n-3})\mathit{\Delta} q = 0 \\
s(p+\mathit{\Delta} p, q+\mathit{\Delta} q) &= s(p,q) + \frac{\partial s}{\partial p}\mathit{\Delta} p + \frac{\partial s}{\partial q}\mathit{\Delta} q \\
&= (b_n + p b_{n-1}) + (-c_{n-1} + b_{n-1} - p c_{n-2})\mathit{\Delta} p + (-c_{n-2} - p c_{n-3})\mathit{\Delta} q \\
&= b_n + (-c_{n-1} + b_{n-1})\mathit{\Delta} p + (-c_{n-2})\mathit{\Delta} q + p\left(b_{n-1} - c_{n-2}\mathit{\Delta} p - c_{n-3}\mathit{\Delta} q\right) = 0 \\
&\therefore \begin{cases} b_{n-1} - \mathit{\Delta} p c_{n-2} - \mathit{\Delta} q c_{n-3} = 0 \\ b_n + \mathit{\Delta} p\left(b_{n-1} - c_{n-1}\right) - \mathit{\Delta} q c_{n-2} = 0 \end{cases} \\
&\therefore \begin{cases} \mathit{\Delta} p = \left(b_{n-1}c_{n-2} - b_n c_{n-3}\right)/z \\ \mathit{\Delta} q = \left\{b_n c_{n-2} + b_{n-1}\left(b_{n-1} - c_{n-1}\right)\right\}/z \\ z = {c_{n-2}}^2 + c_{n-3}\left(b_{n-1} - c_{n-1}\right) \\ \begin{cases} b_i = a_i - p b_{i-1} - q b_{i-2}, b_{-1} = b_{-2} = 0, (\because Eq.(4)) \\ c_i = b_i - p c_{i-1} - q c_{i-2}, c_{-1} = c_{-2} = 0, (\because Eq.(15)) \end{cases} \end{cases}
\end{aligned}
$$

$|\mathit{\Delta} p|, |\mathit{\Delta} q| < \varepsilon$ になるまで求める。

$x^2 + px + q = 0$ より解が求まる。

残った与式は以下となる。

$$
\begin{aligned}
& b_0 x^{n-2} + b_1 x^{n-3} + \cdots + b_{n-3}x + b_{n-2} = 0 && (16) \\
& b_k = a_k - p b_{k-1} - q b_{k-2}, b_{-1} = b_{-2} = 0 && (17)
\end{aligned}
$$

これがプログラミングできれば、低次元化は自動でできる。